"Education to All for Excellence"

## Syllabus

**Definition, Formulation, Solution of PDE ( By Direct Integration Method & Lagranges Method), Non-Linear Partial Differential Equation of First order {Standard I, II, III & IV), Charpit's General Method of Solution**

**PARTIAL DIFFERENTIAL EQUATIONS:**

An equation relating a dependent variable to one or more independent variables by its derivative with respect to the independent variables is called a differential equation.

***Ordinary differential equation (ODE) has only one independent variable while partial differential equation (PDE) has two or more independent variables.***

Examples of ODE and PDE are:

***Examples of some important PDEs:***

(1) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ *One-dimensional wave equation*

(2) $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ *One-dimensional heat equation*

(3) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *Two-dimensional Laplace equation*

(4) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$ *Two-dimensional Poisson equation*

**Formation of partial differential equations :**

**1. By elimination of arbitrary constants:**

**Let $f(x, y, z, a, b) = 0$ be an equation which contains two arbitrary constants '*a*' and '*b*'.**

Therefore partially differentiate the equation w.r.t. $x$ and $y$ to eliminate the constant '*a*' and '*b*'.

**Remark**: *If the number of arbitrary constants is grater than the number of independent variables , then the order of partial differential equation obtained will be more than 1.*

**2. By elimination of arbitrary Functions:** Apply same procedure as above to eliminate arbitrary function.

**Note: we use following notations in PDE**

$$p = \partial z/\partial x \quad q = \partial z/\partial y \quad r = \partial^2 z/\partial x^2 \quad s = \partial^2 z/\partial x \partial y \quad t = \partial^2 z/\partial y^2$$

**Q 1.** From the partial differential equation by eliminating the arbitrary constants from the following

(1) $z=ax+by+ab$ [**Ans:** $z=px+qy+pq$] (2) $z=(x^2+a)(y^2+b)$ [**Ans**: $pq=4xyz$ ]

**Q 2.** From the partial differential equation by eliminating the arbitrary function from the following

(1) $z=f(x^2-y^2)$ (2) $z=y^2+2f(\frac{1}{x}+\log y)$ [**Ans:** $x^2p+yq=2y^2$ **[Dec.03,Jan.06,Dec.10, June12]**

(3) $z=f(x+iy)+g(x-iy)$ **Ans**: $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ [ **Dec. 2003, 2006June 2012, Dec. 2014]**

**Q 3.** Form Partial differential equation $z=e^{xy}f(x-y)$ [**Nov. 2019**]

**Q 4.** Form Partial differential equation $z=f(y/x)$ [**Nov. 2019(o)**]

## Solution of a Partial Differential Equation:

A function *u* is called a solution to a partial differential equation whenever the equation becomes an identity in the independent variables upon substitution of *u* and its appropriate derivatives in the partial differential equation.

Let us consider a Partial Differential Equation of the form $F(x, y, z, p, q)=0$

If it is Linear in $p$ and $q$, it is called a **Linear *Partial Differential Equation*.** (*i.e*. Order and Degree is one)

If it is Not Linear in $p$ and $q$ , it is called as ***nonlinear Partial Differential Equation***(i.e. Order and Degree is other than one)

## Types of solution:

**(a)** A solution in which the number of arbitrary constants is equal to number of independent variables is called **complete integral or complete solution.(** By eliminating the arbitrary constants and from this equation we get $F(x, y, z, p, q)=0$ which is called complete solution ).

**(b)** A solution of $F(x, y, z, p, q)=0$ obtained by giving particular values to $a$ and $b$ in the complete Integral is called a ***particular Integral***

In complete integral if we give particular values to the arbitrary constants we get **particular integral.**

**(c) Singular integral**: let f(x, y, z, p, q) = 0 be a partial differential equation whose complete integral is

$\Phi(x, y, z, a, b) = 0$ …………….. (1)

Differentiating (1) partially w.r.t a and b and then equate to zero, we get

$\partial\Phi/\partial a = 0$ ……………. (2)

$\partial\Phi/\partial b = 0$ ……………. (3)

Eliminate a and b by using (1), (2) and (3).

The eliminate of *a* and *b* is called singular integral.

## LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A Differential Equation which involves partial derivatives and only and no higher order derivatives is called a first order equation. If and have the degree one, it is called a ***linear partial differential equation of first order***; otherwise it is called a ***non-linear partial equation of first order.***

Ex: 1) $yzp + xzq = xy$ is a linear Partial Differential Equation.

2) $x^2p^2 + y^2q^2 = z^2$ is a non-linear Partial Differential Equation

**LAGRANGE'S LINEAR EQUATIONS:** A linear Partial Differential Equation of order one, involving a dependent variable $z$ and two independent variables and $y$, and is of the form

$$\boxed{Pp + Qq = R} \qquad \text{.........(1),}$$

where ***P , Q, R*** are functions of $x, y, z$ is called Lagrange's Linear Equation.

To solve this equation it is enough to solve the **subsidiary/ auxiliary equation**

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}} \qquad \text{.........(2)}$$

If the solution of the subsidiary equation is of the form $u(x, y) = c_1$ and $v(x, y) = c_2$ then the solution of the given Lagrange's equation is $\Phi(u, v) = 0$.

To solve the subsidiary equations we have two methods:

**Method – I (Method of Grouping ) :**

If it is possible to separate variables in eq. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ then, consider any two equations, solve them by Integrating. Take any two members say first two or last two or first and last members. Now consider the first two members $\frac{dx}{P} = \frac{dy}{Q}$. If $P$ and $Q$ contain $z$(other than $x$ and $y$) try to eliminate it. Now direct integration gives $u(x, y) = c_1$. Similarly take another two members $\frac{dy}{Q} = \frac{dz}{R}$ .If $Q$ and $R$ contain $x$(other than $y$ and $z$) try to eliminate it.

Now direct integration gives $v(y, z) = c_2$. Therefore solution of the given Lagrange's equation is $\Phi(u, v) = 0$.

**Q 5.** $yzp + xzq = xy$ **[June 15]**

[Hint: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$ take two pairs and solve Ans: $\Phi(x^2-y^2, y^2-z^2) = 0$]

**Q 6.** Solve $y^2zp + x^2zq = xy^2$ **Ans:** $f(x^3-y^3, x^2-z^2)=0$ **[ DEC. 2005,2007,JUNE 2008, APRIL 2009, Dec. 2010,11,13, June 2015]**

[Hint: $\Phi\left(x^3 - y^3, x^2 - z^2\right) = 0$ take first and last two pairs and solve Ans: $\Phi\left(x^3 - y^3, x^2 - z^2\right) = 0$]

**Q 7.** Solve $xp+yq=3z$ **Ans:** $f(\frac{x}{y},\frac{x^3}{z})=0$ **[ June 2014]**

**Q 8.** Solve $yzp+xzq=xy$ **Ans:** $f(x^2-y^2, y^2-z^2)=0$ **[RGPV DEC. 2006, June 2015,June17]**

**Q 9.** Solve $yq-xp=z$ **Ans:** **[RGPV Nov. 18]**

## Method-II (Method of Substitution):

In some problems, it is possible to solve any two of the equations $\frac{dx}{P}=\frac{dy}{Q}$ (or) $\frac{dy}{Q}=\frac{dz}{R}$ (or) $\frac{dz}{R}=\frac{dx}{P}$ In such cases , solve the differential equation, get the solution and then substitute in the other differential equation.

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**Q 10.** Solve $p+3q=5z+\tan(y-3x)$ **Ans:** $f(y-3x, e^{-5x}\{5z+\tan(y-3x)\})=0$

**Q 11.** Solve $pz-qz=z^2+(x+y)^2$ **[ DEC. 2005, JUNE 2007, DEC. 2010June 2011, 2012]**

**Q 12.** Solve $pzx-qyz=y^2-x^2$

**Q 13.** Solve $x^2p+y^2q=nxy$ **Ans:** $f(\frac{1}{y}-\frac{1}{x}, z-\frac{nxy}{y-x}\log\frac{y}{x})=0$ **[Hint**: sol. $\frac{1}{y}-\frac{1}{x}=c_1$ use it for second sol]

## Method-III ( Method of multiplier's)

Choose any three multipliers *l, m, n* may be constants or function of *x, y* and *z* such that in

$$\boxed{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{l\,P+m\,Q+n\,R}=\frac{l\,dx+m\,dy+n\,dz}{0}=k}$$

In this, we have to choose *l, m, n* so that denominator = 0. That will give us solution by integrating $l\,dx+m\,dy+n\,dz$

the expression $lP + mQ + nR = 0 \Rightarrow ldx + mdy + ndz = 0$ Now direct integration gives ***u(x, y, z) = c₁***.
similarly choose another set of multipliers $l', m', n'$ such that

$$\boxed{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l'dx+m'dy+n'dz}{l'P+m'Q+n'R}=\frac{l'dx+m'dy+n'dz}{0}}$$

$l'dx + m'dy + n'dz = 0$ (as explained earlier),Now direct integration gives $\mathbf{v(x, y, z) = c_2}$.

Therefore solution of the given Lagrange's equation is $\boldsymbol{\Phi(u, v) = 0}$.

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**Q 14.** Solve $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$ **Ans** : $f(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}, xyz)=0$ **.[RGPV DEC. 2008]**

**[ Hint: use** $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \Rightarrow \frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}$ taking $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$ and $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ as multipliers

**Q 15.** Solve $x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$ Ans : $f\left(x^2+y^2-2z, xyz\right)=0$ **[Nov. 2019(o)]**

[ **Hint: use** ***x,y,-1*** and $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ as multipliers]

**Q 16.** Solve $x(y-z)p+y(z-x)q=z(x-y)$ **Ans** : $f\left(x+y+z, xyz\right)=0$ **[ RGPV JUNE. 2007]**

[Hint: use 1,1,1 and $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ as multipliers]

**Q 17.** Solve $(x-y)p+(x+y)q=2xz$ **[Nov. 2019]**

**Q 18.** Solve $(\frac{y-z}{yz})p+(\frac{z-x}{zx})q=\frac{x-y}{xy}$ **Ans** : $f\left(x+y+z, xyz\right)=0$ **[ RGPV JUNE2007, 2009]**

**[Hint : multiply the eq. by** ***xyz*** **then it reduce to previous qus.]**

**Q 19.** Solve $(mz-ny)p+(nx-lz)q=ly-mx$ **Ans :** $f(x^2+y^2+z^2, lx+my+nz)=0$

**[ Hint: use** $x,y,z$ and $l,m,n$ as multipliers]

**Find one solution by linear / homogenous eq.:**

**Q 20.** Solve $(z^2-2yz-y^2)p+(xy+zx)q=xy-zx$ **Ans:** $f(x^2+y^2+z^2, y^2-z^2-2yz)=0$

**[ JUNE , DEC. 2004, Dec. 2014]**

[Hint : $\frac{dx}{(z^2-2yz-y^2)}=\frac{dy}{(xy+zx)}=\frac{dz}{xy-zx}=\frac{xdx+ydy+zdz}{0}$ and solve $\frac{dy}{(xy+zx)}=\frac{dz}{xy-zx}$ using **homogenous eq.** , ]

**Q 21.** Solve $(z^2+y^2-x^2)p-2xyq+2zx=0$ **Ans:** $f(\frac{x^2+y^2+z^2}{z},\frac{y}{z})=0$ **[June04,feb06,dec14]**

**Q 22.** Solve $y^2p-xyq=x(z-2y)$ **Ans :** $f(x^2+y^2, yz-y^2)=0$ **[ JUNE 2003, DEC. 2005, June 2011]**

Hint:

**Method IV:** Add or Subtract two or more terms to make integrable pairs:

**Q 23.** Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$ **[Hint** $\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}=\frac{dz-dx}{z-x}$ **]Ans** $f(\frac{x-y}{y-z},\frac{y-z}{z-x})=0$

**[ DEC. 2002, 2003, JUNE 2007, 2008,June 2013]**

**Q 24.** Solve $(y+z)p+(z+x)q=(x+y)$ [Hint: $\frac{dx-dy}{-(x-y)}=\frac{dy-dz}{-(y-z)}=\frac{dx+dy+dz}{2(x+y+z)}$ **RGPV SEP. 2009]**

**Q 25.** Solve $x^2p+y^2q=(x+y)z$ **Ans** : $f\left(\frac{x-y}{xy},\frac{xy}{z}\right)=0$ **[ Feb. 2005, 2010]**

[Hint: find first sol. $\frac{1}{x}-\frac{1}{y}=c_1$ from first pairs, and **use** $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ as multipliers for second sol.]

**Q 26.** Solve $(x^2-y^2-z^2)p+2xyq=2xz$ **Ans** : $f\left(x^2+y^2+z^2,\frac{y}{z}\right)=0$

[Hint: $\frac{dy}{y}=\frac{dz}{z}$ and $\frac{xdx+ydy+zdz}{-x(x^2+y^2+z^2)}=\frac{dz}{-2xz}$] **[ RGPV JUNE 2004,JAN. 2006, Dec. 2007, Dec. 2014]**

**Q 27.** Solve $(y+z)p+(z+x)q=x+y$ **[ sep.2009]** [Hint: $\frac{dx+dy+dz}{2(x+y+z)}=\frac{dx-dy}{-(x-y)}=\frac{dy-dz}{-(y-z)}$ ]

Ans: $\phi\left(\frac{x-y}{y-z},(x-y)^2(x+y+z)\right)=0$

## FIRST ORDER AND HIGHER DEGREE PARTIAL DIFFERENTIAL EQUATIONS:

### NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS of ORDER FIRST :

**Type 1: Standerd form –I : *f (p, q) = 0***

Equations of the type $f(p, q) = 0$ i.e. equations containing $p$ and $q$ only

Let the required solution be $z= ax+by+c$

Since $\frac{\partial z}{\partial x}=a$ and $\frac{\partial z}{\partial y}=b$

Substituting these values in $f(p, q) = 0$, we get $f(a,b) = 0$

From this, we can obtain $b$ in terms $a$ of (or) $a$ in terms of $b$

Let then the required solution is $\boxed{z=ax+\phi(a)y+c}$

**Note:** Since, the given equation contains two first order partial derivatives, the final Solution should contain only two constants.

**Q 28.** Solve $p^2+q^2=m^2$ ***[Ans: Complete Sol.:*** $z=ax+\sqrt{m^2-a^2}\,y+c$ , ***General Sol*** $z=ax+\sqrt{m^2-a^2}\,y+\varphi(a)$ *diff. wrt. "a" and on elimination of "a ". Singular sol not exist for standard form –I]*

**Q 29.** Solve $q=pq+p^2$ **[ Ans: *Complete Sol.:*** $z=ax+\frac{a}{1-a}y+c$ , $a\neq 0$,

**Q 30.** Solve $p^2+q^2=npq$ ***[Ans: Complete Sol.:*** $z=ax+\frac{a}{2}[n\pm\sqrt{n^2-4}\,]y+c$,

**Type 2: Standerd form –II : *f (z, p, q) = 0***

Let us consider the equation of the type $f(z, p, q) = 0$ ………………..(1)

Let $z$ is a function of $u$ and $u=x+ay$ *i.e.* $z=z(u)$ and $u= x+ay$

Now $\boxed{p=\frac{\partial z}{\partial x}=\frac{dz}{du}.\frac{\partial u}{\partial x}=\frac{dz}{du}.1=\frac{dz}{du} \text{ and } q=\frac{\partial z}{\partial y}=\frac{dz}{du}.\frac{\partial u}{\partial y}=\frac{dz}{du}.a=a\frac{dz}{du}\ [u=x+ay,\frac{\partial u}{\partial a}=1 and \frac{\partial u}{\partial y}=a]}$

Then by (1) $\boxed{f(\frac{\partial z}{\partial u},a\frac{\partial z}{\partial u},z)=0}$ is 1st order differential equation in terms of dependent variable $z$ and independent variable $u$.

Solve this differential equation and finally substitute $u=x+ay$ gives the required solution.

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**Q 31.** Solve $p^2=zq$ [Ans: $z=be^{ax+a^2y}$ ]

**Q 32.** Solve $9(p^2z+q^2)=4$ [Ans: $(z+a^2)^{3/2}=(x+ay+b)$ ]

**Q 33.** Solve $z^2=p^2+q^2$ **Ans:** $(x+ay+b)^2=4z(1+a^2)$ **[ JUNE 2011]**

**Q 34.** Solve $p^3+q^3=27z$ **Ans:** $(1+a^3)z^2=8(x+ay+b)^3$ **[ June 2013].**

**Q 35.** Solve $p(1+q)=zq$ [Ans: $az-1=be^{x+ay+b}$ ] **[June12]**

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## Type 3: Standerd form –III : $f_1(p,x)=f_2(y,q)$

Let us consider the differential equation is of the form $f_1(p,x)=f_2(y,q)$

Let $f_1(p,x)=f_2(y,q)=a$ (say)

Now , writing $p$ in terms of $x$ and $q$ in terms of $y$.

Now, $f(P,Q)=0$ , where $p$ is in terms of $x$ and $q$ is in terms of $y$

From $\int dz=\int p\,dx+\int q\,dy+c$ , we get the required complete solution.

Note: This method is used only when it is possible to separate variables.

**Q 36.** Solve $pe^y=qe^x$ **Ans:** $z+b=a(e^x+e^y)$

**Q 37.** Solve $p^2+q^2=x+y$ Ans: $z+b=\frac{2}{3}\left[(x+a)^{3/2}+(y-a)^{3/2}\right]$

**Q 38.** Solve $q=px+q^2$ **Ans:** $z=a\log x+\frac{1\pm\sqrt{1-4a}}{2}y+b$ **[ June 2011]**

**Q 39.** Solve $q-p=y-x$

**Q 40.** Solve $p^2-q^2=x-y$ Ans: $z=\frac{2}{3}\left[(x+a)^{3/2}+(y+a)^{3/2}\right]+b$ [**June 03,08,09]**

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## Type 4: Standard form –IV : $z=px+qy+f(p,q)$ [ *Caliraut's equation*]

Equation of the form $z=ax+by+c$ then $\frac{\partial z}{\partial x}=a \Rightarrow p=a$ and $\frac{\partial z}{\partial y}=b \Rightarrow q=b$

Hence required solution is ***z = ax +by + f(a,b), i.e directly put p=a and q=b in the given equation.***

**Singular Solution:** Differentiate complete solution partially w.r.t. a and b and eliminate a and b to get singular solution.

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**Q 41.** Solve $z=px+qy+\log pq$ [**Ans**: Complete Sol. $z=ax+by+\log ab$ for Singular sol. $a=\frac{-1}{x}, b=\frac{-1}{y}$ ]

**Q 42.** Solve $z=px+qy+p^2+q^2$ [**Ans**: Complete Sol. $z=ax+by+a^2+b^2$ for Singular sol. $a=\frac{-x}{2}, b=\frac{-y}{2}$ ]

**Q 43.** Solve $z=px+qy+pq$ [Ans: Complete Sol. $z=ax+by+ab$ for Singular sol. $a=-y, b=-x$ ]

**Q 44.** Solve $(p-q)(z-px-qy)=1$ **Ans**: Complete Sol. $z=ax+by+\frac{1}{a-b}$ **[Dec.2007]**

**Q 45.** Solve $(px+qy-z)^2=1+p^2+q^2$ **Ans**: Complete Sol. $z=ax+by\pm\sqrt{1+p^2+q^2}$

**Q 46.**

## Type 5: Equations Reducible to Standard Form:

**Q 47.** Solve $x^2p^2+y^2q^2=z^2$ **Ans:,** $\log z=a\log x+\sqrt{1-a^2}\log y+c_1$ **[Hint: Reduce in form –I]**
**[ RGPV DEC.. 2002, Feb.2005, June 2008, Feb. 2010, Dec.2011,2012,June17]**

**Q 48.** Solve $(x+y)(p+q)^2+(x-y)(p-q)^2=1$ **Ans:,** $z=a\sqrt{(x+y)}+\sqrt{1-a^2}\sqrt{(x-y)}+c_1$

[**Hint: Reduce in form –I, Put** $x+y=X^2$ , $x-y=Y^2$**]** **[RGPV. JUNE 2006]**

**Q 49.** Solve $(y-x)(qy-px)=(p-q)^2$ **Ans:,** $z=b^2(x+y)+bxy+c$

[**Hint: Reduce in form –I, Put** $X=x+y$ , $Y=xy$**]** **[RGPV]**

**Q 50.** Solve $z^2(p^2+q^2)=x^2+y^2$

[**Hint: Reduce in form –III,** $\left(\frac{z}{1}\frac{\partial z}{\partial x}\right)^2+\left(\frac{z}{1}\frac{\partial z}{\partial y}\right)^2=x^2+y^2$, Put zdz=dZ , dx=dX , dy=dY

then it becomes $P^2+Q^2=x^2+y^2$ ]

**Ans:** $z^2=x\sqrt{(a+x^2)}+a\log\{x+\sqrt{(a+x^2)}\}+y\sqrt{y^2-a}-a\log\{y+\sqrt{y^2-a}\}+b$ **[June 2003, 2008, 2009]**

**Q 51.** Solve $z(p^2-q^2)=x-y$ **[Nov. 2019 (o)]**

[**Hint: Reduce in form –III,** $\left(\frac{\sqrt{z}}{1}\frac{\partial z}{\partial x}\right)^2-\left(\frac{\sqrt{z}}{1}\frac{\partial z}{\partial y}\right)^2=x-y$, Put $\sqrt{z}dz=dZ$ , dx=dX

dy=dY then it becomes $P^2-Q^2=x-y$ ]**Ans:** $z=\frac{2}{3}(x+a)^{3/2}+\frac{2}{3}(y+a)^{3/2}+b$

**Q 52.** Solve $2(z+xp+yq)=yp^2$

**CHARPIT'S METHOD:** This is a general method to find the complete integral of the non-linear PDE of the form *f(x,y,z,p,q)*=0 . The Auxiliary Equations are given by

$$\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dp}{p\frac{\partial f}{\partial z}+\frac{\partial f}{\partial x}}=\frac{dq}{q\frac{\partial f}{\partial z}+\frac{\partial f}{\partial y}}$$

Here we have to take the terms whose integrals are easily calculated, so that it may be easier to solve $p$ and $q$ .
Finally substitute in the equation *dz=p dx +q dy* . Integrate it, we get the required solution.

**Q 53.** Using the Charpit's method Solve $z=px+qy+p^2+q^2$ **[ RGPV JUNE 2001]**

**[Hint:** $\frac{dx}{-(x+2p)}=\frac{dy}{-(y+2q)}=\frac{dz}{-p(x+2p)-q(y+2q)}=\frac{dp}{0}=\frac{dp}{0}$

solve last pairs $\frac{dp}{0}=\frac{dq}{0}$ we get $\boxed{\text{p=a, and q=b}}$ **Put this value in question find q in terms of x and y , i.**e. Complete sol.
$\int dz=\int pdx+\int qdy+c\Rightarrow$

**Q 54.** Using the Charpit's method Solve $(p^2+q^2)y=qz$ Ans: $z^2-a^2y^2=(ax+b)^2$
**[ JUNE 2007,2012,Dec. 2012,2013,June 2015]**

**Q 55.** Using the Charpit's method Solve $px+qy=pq$ **[ RGPV JAN 2007, june17 ]**

[Hint: $\frac{dx}{-(x+q)}=\frac{dy}{-(y+p)}=\frac{dz}{-px-pq-qy-qp}=\frac{dp}{p}=\frac{dp}{q}$

solve last pairs $\frac{dp}{p}=\frac{dq}{q}$ we get p=aq

Put this value in question find q in terms of x and y , i.e. $q=-\frac{ax+y}{a}$ and the $p=aq=-a(\frac{ax+y}{a})=-(ax+y)$

Complete sol. $\int dz=\int pdx+\int qdy+c \Rightarrow z=e^{xy}f(x-y)$ or $adz=-(ax+y)adx-(ax+y)dy$

**Q 56.** Using the Charpit's method Solve $2z+p^2+qy+2y^2=0$ **Ans:** $z=(ax+b)y^{-2}+\frac{1}{2}(a^2-y^2)y^{-4}$
**[June 2006]**

**Q 57.** Using the Charpit's method Solve $p^2+qy=z$

**Q 58.** Using the Charpit's method Solve $2(z+xp+yq)=yp^2$ **Ans:** $z=\frac{ax}{y^2}-\frac{a^2}{4y^3}+\frac{b}{y}$ **[June 2006]**

**Q 59.** Using the Charpit's method $z=pq$

**Q 60.** Solve by charpit's method $q=3p^2$ [**Nov. 2019**]

## LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS :

Linear 2nd-Order PDE's

Let u is the dependent variable and x and y are the independent variables

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D=0$$

u(x,y), A(x,y), B(x,y), C(x,y), and D(x,y,u,$\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$)

**where**

The PDE is nonlinear if A, B, or C include u, ∂u/∂x, or ∂u/∂y, or if D is nonlinear in u and/or its first derivatives.

**Classification**: Similar to the technique used to obtain an analytical solution,

$B^2 - 4AC < 0$ ——> Elliptic (e.g. Laplace Eq.)
$B^2 - 4AC = 0$ ——> Parabolic (e.g. Heat Eq.)
$B^2 - 4AC > 0$ ——> Hyperbolic (e.g. Wave Eq.)

## . Solution of Homogeneous Linear PDE with constant coefficients:

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^{n-1} z}{\partial x^{n-1}\partial y} + .................. + A_n \frac{\partial^2 Z}{\partial x^{n-2}\partial y^2} + ............... + A_n \frac{\partial^n z}{\partial y^n} = f(x,y)$$

Where $A_0, A_1 \ldots\ldots A_n$ are constants.

The **General solution** of above equation is $\boxed{z = z_c + z_p = C.F. + P.I.}$

**Step-I : To find Complementary function( Integral)**

**Change the given equation in D-Notation by putting**

$$\frac{\partial}{\partial x} = D,\ \frac{\partial}{\partial y} = D',\ \frac{\partial^2}{\partial x^2} = D^2,\ \frac{\partial^2}{\partial y^2} = D'^2,\ \frac{\partial^2}{\partial x \partial y} = DD',\ \frac{\partial^3}{\partial x^2 \partial y} = D^2 D',\ \frac{\partial^3}{\partial x \partial y^2} = DD'^2$$

Form auxillary equation by putting $\boxed{D = m,\ D' = 1}$

**If roots of auxillary equation are (i) Distinct , say $m_1$ ,$m_2$ then** $C.F. = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$

**If roots of auxillary equation are (ii) Equal , say $m_1$=$m_2$=m then** $C.F. = \phi_1(y + m\ x) + x\phi_2(y + mx) + ....$

Note: There is no separate rule for imaginary roots.

**Step-II : To find P.I.( Particular integral/ Solution)**

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| When *f(x,y)*= $e^{a+byx}$ | Put $D = a$ , and D'=b when *F(D,D')=F(a,b)* ≠*0* | $P.I. = \frac{1}{F(D,D')} e^{ax+by} = \frac{e^{ax+by}}{F(a,b)}$ |

**Q 61.** Solve $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\ \partial y} + \frac{\partial^2 z}{\partial y^2} = e^{3x+2y}$ **[June 2015]**

**Q 62.** Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x+2y}$ **[Nov. 2018]**

**Exceptional case:**

| | | |
|---|---|---|
| When *f(x,y)*= $e^{a+byx}$ **(Exceptional case** When *F(D,D')=F(a,b)* =0 | Put *D=D+a* and *D'=D'+b* and Solve the equation for 1= $x^0$ or $e^{0x+0y}$ | $P.I. = \frac{1}{F(D,D')} e^{ax+by} = e^{ax+by} \frac{1}{F(D+a, D'+b)} 1$ |

**Q 63.** Solve $(D^3 - 4D^2 D' + 5DD'^2 - 2D'^3)z = e^{2x+y}$ [Ans: $z = \phi_1(y+1x) + x\phi_2(y+x) + \phi_3(y+2x) + \frac{x^5}{20}$ **[June03,08,16]**

**Q 64.** Solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x\ \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$ [Ans: $z = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{x^2}{2} e^{2x+y}$ ]**[Dec.11]**

**Q 65.** Solve $(D^2 - 2DD' + D'2)z = e^{x+y}$ **[ RGPV June . 2007]**

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| **When** ***f(x,y)= sin(ax+by) or cos(ax+by)*** | Put $D^2$=-($a^2$ ), $D'^2$=-($b^2$) , *DD'=-ab* and Solve the equation for *D* or *D'* by rationalization of the equation (same for *cosax*) Except *f*($-a^2$, $-b^2$) ≠ 0 | $P.I. = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax+by)$ |

**Q 66.** Find the particular integral of $\frac{\partial^3 z}{\partial x^3}-7\frac{\partial^2 z}{\partial x\,\partial y^2}-6\frac{\partial^2 z}{\partial y^3}=sin(\,x+2y\,)$ [**June 2014]**

**Q 67.** Solve $r+t=\cos mx\,\cos\,ny$

**Q 68.** Solve $(D^2-DD')z=\sin x\cos 2y$ **[ RGPV Dec. 2005, 2007,Dec. 13, Nov. 19 (o)]**

**Q 69.** Solve $(D^2-DD')z=\cos x\cos 2y$ **[ RGPV Dec. 2008]**

**Q 70.** Solve $(D^2-DD')z=\cos(x+2y)$

**Q 71.** Find P.I. $(D^3-7DD'^2-6D'^3)z=\sin(x+2y)$ [Ans: $z=\frac{1}{1575}[27\cos(x+2y)-48\cos(x+2y)]$ **[ June . 14]**

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| When $f(x,y)=x^m y^n$ | Expand Series $F(D/D')^{-1}$ or $F(D'/D)^{-1}$ | using $(1-x)^{-1}=1+x+x^2+x^3$...... $(1+x)^{-1}=1-x+x^2-x^3$...... |
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**Q 72.** Solve $(D^2+3DD'+2D'^2)z=24xy$ [Ans: $z=\phi_1(y-x)+\phi_2(y-2x)+4x^3y-3x^4$ ]**[Dec.03]**

**Q 73.** Solve $(D^2-2DD'+D'^2)z=12xy$ [Ans: ]**[June17]**

**Q 74.** Solve $(D^2-DD'-6D'^2)z=xy$ [Ans: ] **[Nov. 2019]**

**Q 75.** Solve $(D^3-D'^3)z=x^3y^3$ [Ans: $z=\phi_1(y+1x)+\phi_2(y+wx)+\phi_3(y+w^2x)+\frac{x^6y^3}{120}+\frac{x^9}{1080}$ **[ Dec .12 ]**

**Q 76.** Solve $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}-6\frac{\partial^2 z}{\partial y^2}=x+y$ **[Nov.18]**

**Q 77.** Solve $(D^2+3DD'+2D'^2)z=x+y$ **[Nov.2019(o)]**

**Q 78.** Solve $\frac{\partial^2 z}{\partial x^2}+3\frac{\partial^2 z}{\partial x\,\partial y}+2\frac{\partial^2 z}{\partial y^2}=x-y$ [Ans: $z=\phi_1(y-x)+\phi_2(y-2x)+\frac{1}{6}[4x^3-3x^2y]$ **[June12]**

**Q 79.** Solve $(D^2-6DD'+9D'^2)z=12x^2+36xy$

**Q 80.** Solve $(D^2+2DD'+D'^2)z=x^2+y^2+xy$ **[ RGPV Sep.. 2009]**

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**Q 81.** Solve $(D^3+D^2D'-DD'^2-D'^3)z=e^{2x+y}+\cos(x+y)$ **[ RGPV Jan . 2007]**

**Q 82.** Solve (D-D'-1)(D-D'-2)z= $e^{3x-y}$ + x **[ RGPV June 2014]**

**Short Methods for Finding P.I.( Only for Homogeneous PDE)**

$P.I.=\frac{1}{F(D,D')}\phi(ax+by)=\frac{1}{F(a,b)}\iiint \int\phi(ax+by)dv^n$ where $v=ax+by$ and $n$=order of diff. equation.

{ Note : This method can be apply for all types of functions, using the above formula}

**Exceptional case: if $F(a,b)=0$**, Factorize the equation in the form of( $bD-aD'$) and use formula

$$P.I.=\frac{1}{(bD-aD')^n}\phi(ax+by)=\frac{x^n}{b^n.\,n!}\phi(ax+by)$$

**Q 83.** $r-t=x-y$ **Ans**: $z=\phi_1(y-x)+\phi_2(y+x)+\frac{x}{4}(x-y)^2$ ]

**Q 84.** $2r-s-3t=\frac{5e^x}{e^y}$ **Ans**: $z=\phi_1(y+\frac{3}{2}x)+\phi_2(y-x)+\frac{x^1}{(-1)^1 1!}e^{x-y}$ ]

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|---|---|---|
| **General Formula** | Put $y=c-mx$ , where $c$ is a constant | $\frac{1}{(D-mD')}f(x,y)=\int f(x,c-mx)dx$ |

**Q 85.** Solve $(D^2+DD'-6D'^2)z=y\cos x$ [**Ans**: $z=\phi_1(y-3x)+\phi_2(y+2x)-y\cos x+\sin x$ ]

[**Hint:** $P.I.=\frac{1}{(D+3D')(D-2D')}y\cos x=\frac{1}{(D+3D')}\left(\frac{1}{(D-2D')}y\cos x\right)$ $P.I.=\frac{1}{(D+3D')}\left(\int(c-2x)\cos x\,dx\right)=\frac{1}{(D+3D')}(y\sin x-2\cos x\,dx)$,
again apply same formula for remaining factor

**[ RGPV Dec. 2002, June 2004 ,June . 2006,09,11,13]**

**Q 86.** Solve $r-s-2t=(y-1)e^x$ **[ RGPV Jan.2006, Feb.. 2010]**

**Q 87.** Solve $(D^2+DD'-2D'^2)z=\sqrt{2x+y}$ **[ Ans**: $z=\phi_1(y+x)+\phi_2(y-2x)+\frac{1}{15}(y+2x)^{3/2}$

[**Hint:** $P.I.=\frac{1}{(D+2D')(D-D')}\sqrt{2x+y}=\frac{1}{(D+2D')}\left(\frac{1}{(D-D')}\sqrt{2x+y}\right)$ $P.I.=\frac{1}{(D+2D')}\left(\int(c+x)^{1/2}dx\right)=\frac{1}{(D+2D')}\left(\frac{(c+x)^{3/2}}{3/2}\right)$,

**Q 88.** Solve $r-2s+t=\tan(y+x)$ **[Ans:** $z=\phi_1(y+x)+x\phi_2(y+x)+\frac{1}{2}x^2\tan(x+y)$

**Q 89.** Solve $2r-5s+2t=5\sin(2x+y)$

[Hint: $f(-a^2,-b^2)=0$ hence use general formula $P.I.=\frac{1}{(2D-D')(D-2D')}\sin(2x+y)=\frac{1}{(2D-D')}\left(\frac{1}{(D-2D')}\sin(2x+y)\right)$, put $y=c-mx=c-2x$ [**Ans:** $z=\phi_1(2y+x)+\phi_2(y+2x)-\frac{5}{3}x\cos(y+2x)$